Numerical Solution of Nth- Order Fuzzy Differential Equations by STHWS Method

S. Sekar and S. Senthilkumar

Abstract— In this paper, we have introduced and studied a new technique namely single term Haar wavelet series (STHWS) for getting the solution of Nth – order fuzzy differential equations based on Seikkala derivative with initial value problem [6]. The obtained discrete solutions were compared with exact solutions and Runge-Kutta method based on Centroidal Mean (RKCeM). Error graphs are presented to highlight the efficiency of the STHWS.

Index Terms— Fuzzy differential equations, Haar wavelets, Runge-Kutta method, Runge-Kutta method basec on Centroidal mean, Single term Haar wavelet series.

1 Introduction

Numerical methods to compute solutions of fuzzy differential equations have already been developed and convergence results proven by Abbasbandy [1] and Duraisamy [4 - 5]. However, these methods are no better than 1 st order accurate and 1 st order accuracy holds with additional assumotions. Further, unless solutions are unique, these methods only guarantee that a sub-sequence of the numerical solutions converges. Fuzzy differential equations are a natural way to medel dynamical systems under uncertainty. First order linear fuzzy differential equations are one of the simplest fuzzy differential equations, which appear in many applications [2 – 3]. The concept of a fuzzy derivative was first introduced by S. L. Chang and L. A. Zadeh.

STHWS plays an important role in both the analysis and numerical solution of differential inclusions. STHWS can have a significant impact on what is considered a practical approach and on the types of problems that can be solved. However, working with fuzzy differential equations places special demands on STHWS codes. In science and engineering, fuzzy differential inclusions often have to be solved [7]. Although some cases can be solved analytically, the majority of fuzzy differential inclusions are too complicated to have analytical solutions. Even when analytical solutions can be found, they are not always useful in practice since the computational cost involved is very high [9].

In recent years, there has been an increased interest in several methods were arisen to solve the fuzzy differential inclusions. STHWS can have a significant impact on what is considered a practical approach and on the types of problems that can be solved. S. Sekar and team of his researchers [10 - 16] introduced the STHWS to study the time-varying nonlinear singular systems, analysis of the differential equations of the sphere, to study on CNN based hole-filter template design, analysis of the singular and stiff delay systems and nonlinear singular systems from fluid dynamics, numerical investigation of nonlinear volterra-hammerstein integral equations, to study on periodic and oscillatory problems, and numerical solution of nonlinear problems in the calculus of variations.

In this paper, we have introduces and studied a new technique for getting the solution of fuzzy initial value problem. The organized paper is as follows: In Section 2, we give some basic results on fuzzy numbers and define a fuzzy derivative and a fuzzy integral then the fuzzy initial values is treated in Section 3 using the extension principle of Zadeh and the concept of fuzzy derivative. It is shown that the fuzzy initial value problem has a unique fuzzy solution when f satisfies Lipschitz condition which guarantees a unique solution to the deterministic initial value problem. In Section 4, the STHWS method for solving Nth – order fuzzy differential equations is introduced. In Section 5 the proposed method is illustrated by solving several numerical examples [6], and the conclusion is drawn in Setion 6.

2 Preliminaries

An arbitrary fuzzy number is represented by an ordered pair of functions $\left(\underline{u}(r), \overline{u}(r)\right)$ for all $r \in [0, 1]$, which satisfy the following requirements [12]:

(i) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0, 1],

(ii) $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0, 1],

(iii) $\underline{u}(r) \le \overline{u}(r) \quad \forall r \in [0, 1]$,

let E be the set of all upper semi-continuous normal convex fuzzy numbers with bounded α-level intervals.

Lemma 2.1 Let $\left[\underline{v}(\alpha), \overline{v}(\alpha)\right]$, $\alpha \in (0, 1]$ be a given family of non-empty intervals. If

(i) $\left[\underline{v}(\alpha), \overline{v}(\alpha)\right] \supset \left[\underline{v}(\beta), \overline{v}(\beta)\right]$ for $0 < \alpha \le \beta$,

and

• *Department of Mathematics, Government Arts College (Autonomous), Cherry road, Salem – 636 007, Tamil Nadu, India*
E-mail:sekar_nitt@rediffmail.com

• *Department of Mathematics, AVC College (Autonomous), Mannampandal, Mayiladuthurai – 609 305, Tamil Nadu, India*
E-mail:sskavcc@gmail.com

ISSN 2229-5518

(ii) $\left[\lim_{k\to\infty}\underline{v}(\alpha_k), \lim_{k\to\infty}\overline{v}(\alpha_k)\right] = \left[\underline{v}(\alpha), \overline{v}(\alpha)\right]$,

whenever (α_k) is a non-decreasing sequence converging to $\alpha \in (0, 1]$, then the family $\left[\underline{v}(\alpha), \overline{v}(\alpha)\right]$, $\alpha \in (0, 1]$, represent the α-level sets of a fuzzy number v *in E*. Conversely if $\left[\underline{v}(\alpha), \overline{v}(\alpha)\right]$, $\alpha \in (0, 1]$, are α-level sets of a fuzzy number $v \in E$, then the conditions (i) and (ii) hold true.

Definition 2.2 Let I be a real interval. A mapping $v : I \to E$ is called a fuzzy process and we denoted the α-level set by $[v(t)]_\alpha = \left[\underline{v}(t,\alpha), \overline{v}(t,\alpha)\right]$. The Seikkala derivative $v'(t)$ *of v* is defined by $[v'(t)]_\alpha = \left[\underline{v}'(t,\alpha), \overline{v}'(t,\alpha)\right]$, provided that is a equation defines a fuzzy number $v'(t) \in E$.

Definition 2.3 Suppose u and v are fuzzy sets in E. Then their Hausdroff $D : E \times E \to R_+ \cup \{0\}$,

$$D(u,v) = \sup_{\alpha\in[0,1]} \max\left\{\left|\underline{u}(\alpha) - \underline{v}(\alpha)\right|, \left|\overline{u}(\alpha) - \overline{v}(\alpha)\right|\right\},$$

i.e.,D(u, v) is maximal distance between α-level sets of u and v.

3 Fuzzy Initial Value Problem

Now we consider the initial value problem

$$\left.\begin{aligned} &x^{(n)}(t) = \psi\left(t, x, x', \dots, x^{(n-1)}\right), \\ &x(0) = a_1, \dots, x^{(n-1)}(0) = a_n \end{aligned}\right\} \quad (1)$$

where ψ is a continuous mapping from $R_+ \times R^n$ into R and a_i ($0 \le i \le n$) are fuzzy numbers in E. The mentioned n^{th}-order fuzzy differential equation by changining variables $y_1(t) = x(t)$, $y_2(t) = x^1(t)$, . . . , $y_n(t) = x^{(n-1)}(t)$, converts to the following fuzzy system

$$\left.\begin{aligned} &y_1'(t) = f_1(t, y_1, \dots, y_n), \\ &\quad \vdots \\ &y_n'(t) = f_n(t, y_1, \dots, y_n), \\ &y_1(0) = y_1^{[0]} = a_1, \dots, y_n(0) = y_n^{[0]} = a_n, \end{aligned}\right\} \quad (2)$$

where f_i ($1 \le i \le n$) are continuous mapping from $R_+ \times R^n$ into R and $y_i^{[0]}$ are fuzzy numbers in E with α-level intervals

$\left[y_i^{[0]}\right]_\alpha = \left[\underline{y}_i^{[0]}(\alpha), \overline{y}_i^{[0]}(\alpha)\right]$ *for* $i = 1, \dots, n$, *and* $0 < \alpha \le 1$.

We call $y = (y_1, \dots, y_n)^t$ is a fuzzy solution of (2) on an interval I, if

$$\left.\begin{aligned} \underline{y}_i'(t,\alpha) &= \min\left\{f_i(t, u_1, \dots, u_n); u_j \in \left[\underline{y}_j(t,\alpha), \overline{y}_j(t,\alpha)\right]\right\} \\ &= \underline{f}_i(t, y(t,\alpha)), \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} \overline{y}_i'(t,\alpha) &= \max\left\{f_i(t, u_1, \dots, u_n); u_j \in \left[\underline{y}_j(t,\alpha), \overline{y}_j(t,\alpha)\right]\right\} \\ &= \overline{f}_i(t, y(t,\alpha)), \end{aligned}\right\} \quad (4)$$

and $\underline{y}_i(0,\alpha) = \underline{y}_i^{[0]}(\alpha), \overline{y}_i(0,\alpha) = \overline{y}_i^{[0]}(\alpha)$

Thus for fixed α we have a system of initial value problem in R^{2n}. If we can solve it (uniquely), we have only to verify that the intervals, $\left[\underline{y}_j(t,\alpha), \overline{y}_j(t,\alpha)\right]$ define a fuzzy number $y_i(t) \in E$. Now let $\underline{y}^{[0]}(\alpha) = \left(\underline{y}_1^{[0]}(\alpha), \dots, \underline{y}_n^{[0]}(\alpha)\right)^t$ and $\overline{y}^{[0]}(\alpha) = \left(\overline{y}_1^{[0]}(\alpha), \dots, \overline{y}_n^{[0]}(\alpha)\right)^t$, with respect to the above mentioned indicators, system (2) can be written as with assumption

$$\left.\begin{aligned} &y'(t) = F(t, y(t)), \\ &y(0) = y^{[0]} \in E^n \end{aligned}\right\} \quad (5)$$

With assumption $y(t,\alpha) = \left[\underline{y}(t,\alpha), \overline{y}(t,\alpha)\right]$ and $y'(t,\alpha) = \left[\underline{y}'(t,\alpha), \overline{y}'(t,\alpha)\right]$ where

$$\underline{y}(t,\alpha) = \left(\underline{y}(t,\alpha), \dots, \underline{y}(t,\alpha)\right)^t \quad (6)$$

$$\overline{y}(t,\alpha) = \left(\overline{y}(t,\alpha), \dots, \overline{y}(t,\alpha)\right)^t \quad (7)$$

$$\underline{y}'(t,\alpha) = \left(\underline{y}'(t,\alpha), \dots, \underline{y}'(t,\alpha)\right)^t \quad (8)$$

$$\overline{y}'(t,\alpha) = \left(\overline{y}'(t,\alpha), \dots, \overline{y}'(t,\alpha)\right)^t \quad (9)$$

and with assumption

$F(t, y(t,\alpha)) = \left[\underline{F}(t, y(t,\alpha)), \overline{F}(t, y(t,\alpha))\right]$, where

$$\underline{F}(t, y(t,\alpha)) = \left(\underline{f}_1(t, y(t,\alpha)), \dots, \underline{f}_n(t, y(t,\alpha))\right)^t, \quad (10)$$

$$\overline{F}(t, y(t,\alpha)) = \left(\overline{f}_1(t, y(t,\alpha)), \dots, \overline{f}_n(t, y(t,\alpha))\right)^t, \quad (11)$$

$y(t)$ is a fuzzy solution of (5) on an interval I for all $\alpha \in (0, 1]$, if

$$\left.\begin{aligned} &\underline{y}'(t,\alpha) = \underline{F}(t, y(t,\alpha)); \\ &\overline{y}'(t,\alpha) = \overline{F}(t, y(t,\alpha)); \\ &\underline{y}(0,\alpha) = \underline{y}^{[0]}(\alpha), \overline{y}(0,\alpha) = \overline{y}^{[0]}(\alpha) \end{aligned}\right\} \quad (12)$$

or

$$\left.\begin{aligned} &y'(t,\alpha) = F(t, y(t,\alpha)), \\ &y(0,\alpha) = y^{[0]}(\alpha), \end{aligned}\right\} \quad (13)$$

Now we show that under the assumption for functions f_i, for $i = 1, \dots, n$ how we can obtain a unique fuzzy solution for system (2).

Theorem 3.1 If $f_i(t, u_1, \dots, u_n)$ for $i = 1, \dots, n$ are continuous function of t and satiesfies the Lipschitz condition in $u = (u_1, \dots, u_n)^t$ in the region $D = \{(t, u) \mid t \in [0, 1], -\infty < u_i < \infty$ for $i = 1, \dots, n\}$ with constant L_i then the initial value problem (2) has a unique fuzzy solution in each case.

ISSN 2229-5518

Proof. See [8].

4 Single-Term Haar Wavelet Series Method

The orthogonal set of Haar wavelets $h_i(t)$ is a group of square waves with magnitude of ± 1in some intervals and zeros elsewhere [12]. In general,

$$\left.\begin{array}{l} h_n(t) = h_1\left(2^j t - k\right), n = 2^j + k, \\ j \geq 0, 0 \leq k < 2^j, n, j, k \in Z \end{array}\right\}$$

$$h_1(t) = \begin{cases} 1, 0 \leq t < \frac{1}{2} \\ -1, \frac{1}{2} \leq t < 1 \end{cases},$$

Namely, each Haar wavelet contains one and just one square wave, and is zero elsewhere. Just these zeros make Haar wavelets to be local and very useful in solving stiff systems. Any function *y(t),* which is square integrable in the interval [0,1). Can be expanded in a Haar series with an infinite number of terms

$$\left.\begin{array}{l} y(t) = \sum_{i=0}^{\infty} c_i h_i(t), i = 2^j + k, \\ j \geq 0, 0 \leq k < 2^j, n, j, t \in [0,1] \end{array}\right\} \quad (10)$$

where the Haar coefficients

$$c_i = 2^j \int_0^1 y(t) h_i(t) dt$$

are determined such that the following integral square error ε is minimized:

$$\left.\begin{array}{l} \varepsilon = \int_0^1 \left[y(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right]^2 dt, \\ m = 2^j, j \in \{0\} \cup N \end{array}\right\}$$

usually, the series expansion Equation (10) contains an infinite number of terms for a smooth *y(t)*. If *y(t)* is a piecewise constant or may be approximated as a piecewise constant, then the sum in Eq. (10) will be terminated after *m* terms, that is

$$y(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) = c_{(m)}^T h_{(m)}(t), t \in [0,1]$$

$$c_{(m)}(t) = \left[c_0 c_1 \ldots c_{m-1}\right]^T, \quad (11)$$

$$h_{(m)}(t) = \left[h_0(t) h_1(t) \ldots h_{m-1}(t)\right]^T,$$

where "T" indicates transposition, the subscript *m* in the parantheses denotes their dimensions. The integration of Haar wavelets can be expandable into Haar series with Haar coefficient matrix *P[3]*.

$$\int h_{(m)}(\tau) d\tau \approx P_{(m \times m)} h_{(m)}(t), t \in [0,1]$$

where the m-square matrix P is called the operational matrix of integration and single-term $P_{(1\times1)} = \frac{1}{2}$. Let us define [12]

$$h_{(m)}(t) h_{(m)}^T(t) \approx M_{(m \times m)}(t),$$

and $M_{(1\times1)}(t) = h_0(t)$. Equation (3) satisfies

$$M_{(m \times m)}(t) c_{(m)} = C_{(m \times m)} h_{(m)}(t),$$

where $c_{(m)}$ is defined in Equation (11) and $C_{(1\times1)} = c_0$.

5 Numerical Examples

To show the efficiency of the STHWS, we have considered the following problem taken from [6] and [8], with step size $h = 0.1$ along with the exact solutions. The discrete solutions obtained by the two methods, STHWS and the RKCeM methods; the absolute errors between them are tabulated and are presented in Table 1 - 4. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of " x "and are presented in Fig. 1 to Fig. 6 for the following problem, using three dimensional effects.

Example 5.1 Cosider the following fuzzy differential equation with fuzzy initial value [6]

$$\left.\begin{array}{l} y'''(t) - 4y'(t) + 4y(t) = 0, (t \geq 0) \\ y(0) = (2 + \alpha, 4 - \alpha) \\ y'(0) = (5 + \alpha, 7 - \alpha) \end{array}\right\}$$

The exact solution is as follows:

$$\underline{y}(t, r) = (2 + r) e^{2t} + (1 - r) t e^{2t}$$

$$\overline{y}(t, r) = (4 - r) e^{2t} + (r - 1) t e^{2t}$$

Example 5.2 Cosider the following fuzzy differential equation with fuzzy initial value [6][8]

$$\left.\begin{array}{l} y'''(t) = 2y''(t) + 3y'(t), (0 \leq t \leq 1) \\ y(0) = (3 + \alpha, 5 - \alpha) \\ y'(0) = (-3 + \alpha, -1 - \alpha) \\ y''(0) = (8 + \alpha, 10 - \alpha) \end{array}\right\}$$

The eigen value – eigen vector solution is as follows:

$$y(t, r) = \left(-\frac{1}{3} + \frac{7}{12} e^{3t} + \left(\frac{11}{4} + r \right) e^{-t}, -\frac{1}{3} + \frac{7}{12} e^{3t} + \left(\frac{19}{4} - r \right) e^{-t} \right)$$

TABLE 1
EXACT, DISCRETE SOLUTIONS AND ERROR CALCULATION OF EXAMPLE 5.1 FOR $\underline{y}(t,r)$

r	Example 5.1 : $\underline{y}(t,r)$				
	Exact Solutions	RKCeM Solutions	RKCeM Error	STHWS Solution	STHWS Error
0	2.563719	2.564946	1.23E-03	2.56372	1E-06
0.1	2.673767	2.674872	1.10E-03	2.673769	2E-06
0.2	2.783815	2.784798	9.83E-04	2.783818	3E-06
0.3	2.893863	2.894725	8.61E-04	2.893867	4E-06
0.4	3.003912	3.004651	7.39E-04	3.003917	5E-06
0.5	3.113961	3.114577	6.17E-04	3.113967	6E-06
0.6	3.224009	3.224503	4.94E-04	3.224016	7E-06
0.7	3.334058	3.33443	3.71E-04	3.334066	8E-06
0.8	3.444108	3.444356	2.48E-04	3.444117	9E-06
0.9	3.554157	3.554282	1.25E-04	3.554167	1E-05
1	3.664206	3.664208	1.91E-06	3.664217	1.1E-05

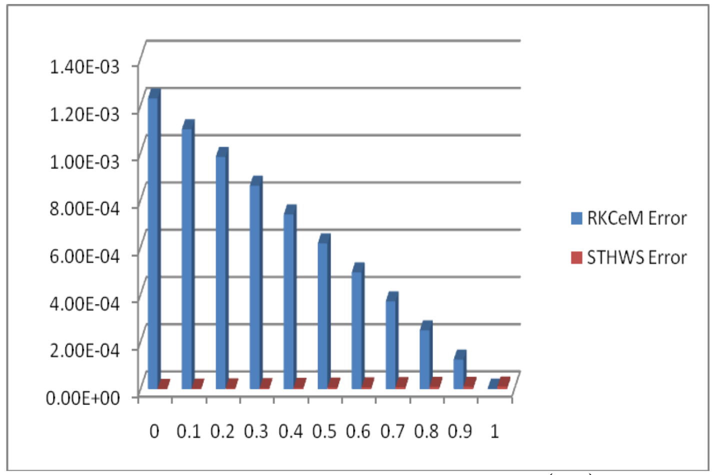

Fig. 2. Error graph for Example 5.1 : $\underline{y}(t,r)$

TABLE 2
EXACT, DISCRETE SOLUTIONS AND ERROR CALCULATION OF EXAMPLE 5.1 FOR $\overline{y}(t,r)$

r	Example 5.1 : $\overline{y}(t,r)$				
	Exact Solutions	RKCeM Solutions	RKCeM Error	STHWS Solutions	STHWS Error
0	4.764708	4.763471	1.24E-03	4.764709	1E-06
0.1	4.654658	4.653545	1.11E-03	4.654659	1E-06
0.2	4.544607	4.543618	9.89E-04	4.54461	3E-06
0.3	4.434557	4.433692	8.65E-04	4.434561	4E-06
0.4	4.324506	4.323766	7.40E-04	4.324511	5E-06
0.5	4.214456	4.21384	6.16E-04	4.214462	6E-06
0.6	4.104406	4.103913	4.92E-04	4.104413	7E-06
0.7	3.994356	3.993987	3.69E-04	3.994364	8E-06
0.8	3.884306	3.884061	2.45E-04	3.884315	9E-06
0.9	3.774256	3.774135	1.21E-04	3.774266	1E-05
1	3.664206	3.664208	1.91E-06	3.664217	1.1E-05

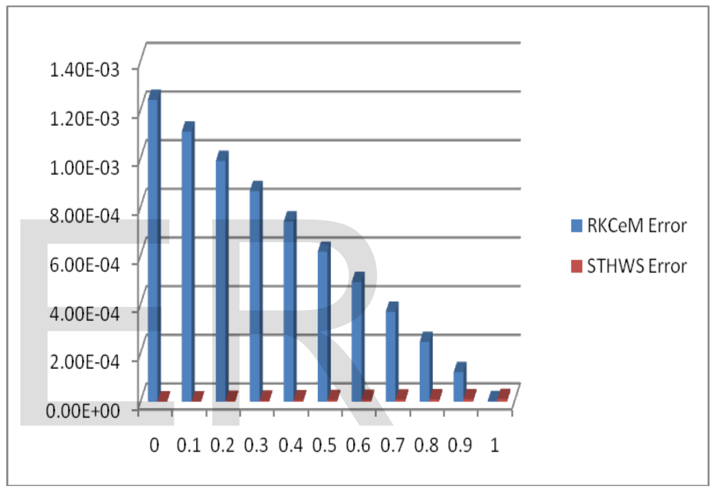

Fig. 3. Error graph for Example 5.1 : $\overline{y}(t,r)$

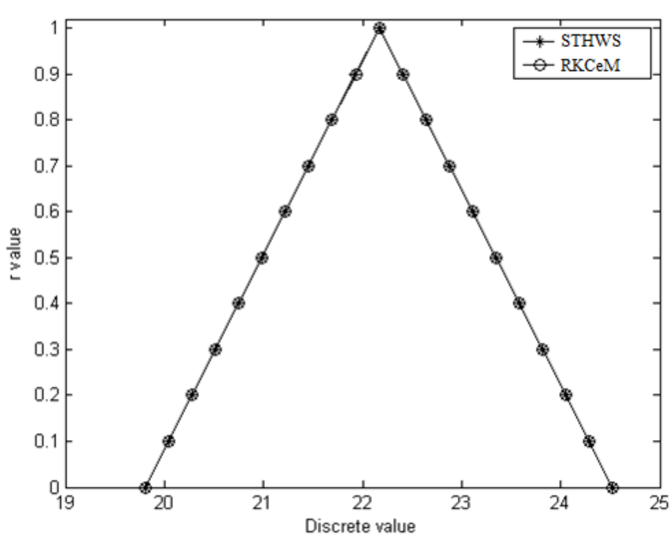

Fig. 1. Solution graph for Example 5.1

TABLE 3
EXACT AND DISCRETE SOLUTIONS OF EXAMPLE 5.2

r	Example 5.2					
	Exact Solutions		RKCeM Solutions		STHWS Solutions	
	$y_1(t_i;r)$	$y_2(t_i;r)$	$y_1(t_i;r)$	$y_2(t_i;r)$	$y_1(t_i;r)$	$y_2(t_i;r)$
0.1	12.4317	13.0939	12.4313	13.0935	12.4317	13.0939
0.2	12.4685	13.0571	12.4681	13.0567	12.4685	13.0571
0.3	12.5053	13.0203	12.5049	13.0199	12.5053	13.0203
0.4	12.5421	12.9835	12.5417	12.9831	12.5421	12.9835
0.5	12.5788	12.9467	12.5785	12.9463	12.5788	12.9467
0.6	12.6156	12.9099	12.6152	12.9095	12.6156	12.9099
0.7	12.6524	12.8731	12.6520	12.8728	12.6524	12.8731
0.8	12.6892	12.8364	12.6888	12.8360	12.6892	12.8364
0.9	12.7260	12.7996	12.7256	12.7992	12.7260	12.7996
1.0	12.7628	12.7628	12.7624	12.7624	12.7628	12.7628

TABLE 4

Error Calculation of Example 5.2

r	Example 5.2			
	RKCeM Error		STHWS Error	
	$y_1(t_i;r)$	$y_2(t_i;r)$	$y_1(t_i;r)$	$y_2(t_i;r)$
0.1	0.0004	0.0004	0.0002	0.0002
0.2	0.0004	0.0004	0.0002	0.0002
0.3	0.0004	0.0004	0.0002	0.0002
0.4	0.0004	0.0004	0.0002	0.0002
0.5	0.0003	0.0004	0.0002	0.0002
0.6	0.0004	0.0004	0.0002	0.0002
0.7	0.0004	0.0003	0.0002	0.0002
0.8	0.0004	0.0004	0.0002	0.0002
0.9	0.0004	0.0004	0.0002	0.0002
1.0	0.0004	0.0004	0.0002	0.0002

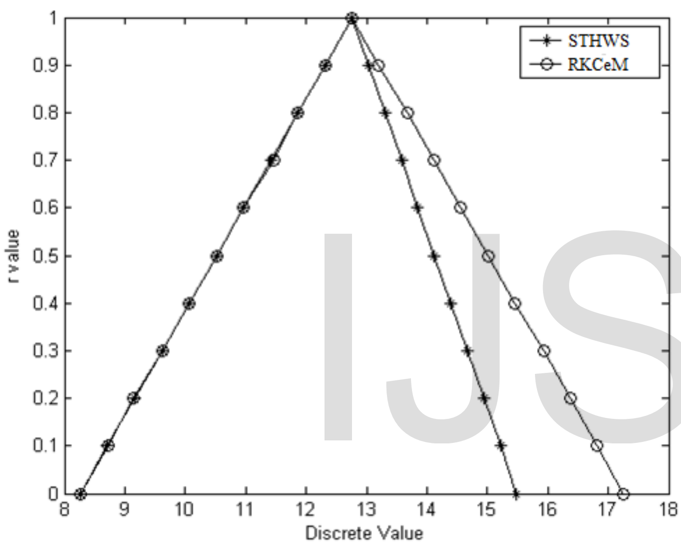

Fig. 4. Solution graph for Example 5.2

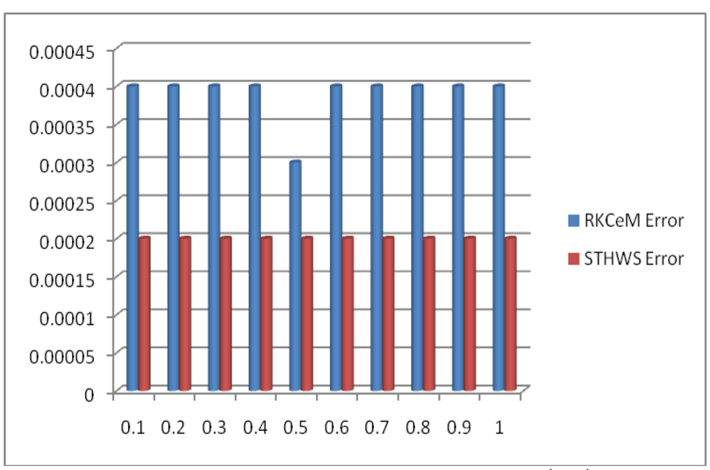

Fig. 5. Error graph for Example 5.2 : $y_1(t_i;r)$

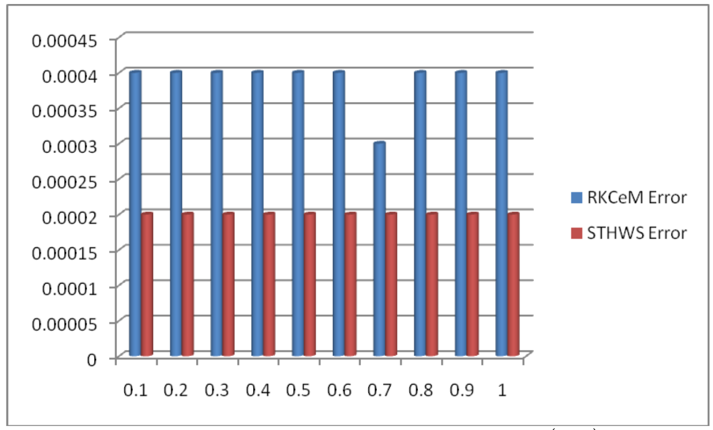

Fig. 6. Error graph for Example 5.2 : $y_2(t_i;r)$

6 Conclusion

In this paper, a new numerical method for solving Nth - order fuzzy initial value problem is proposed. Here the Nth - order fuzzy linear differential equation is converted to a fuzzy system which will be solved with the STHWS. From the numerical examples, we could conclude that the proposed method almost coincides with the exact solution and the classical fourth order Runge – Kutta method (refer Table 1 - 4 and Figure 1 - 6).

Acknowledgment

The authors gratefully acknowledge the Dr.A.R.Rajamani, Principal, Government Arts College (Autonomous), Salem-636 007, for encouragement and support. The authors also thank Mr.R.P.Sampathkumar, Associate Professor and Head of the Department of Mathematics, Government Arts College (Autonomous), Salem-636 007, Tamil Nadu, India, for his kind help and encouragement.

References

[1] S. Abbasbandy and T. Allahviranloo, "Numerical solutions of fuzzy differential equations by Taylor method", *Journal of Computational Methods in Applied Mathematics.* vol.2, pp. 113-124, 2002.

[2] J.J. Buckley and E. Eslami, *Introduction to Fuzzy Logic and Fuzzy Sets*, Physica- Verlag, Heidelberg, Germany, 2001.

[3] J.J. Buckley, E. Eslami and T. Feuring, *Fuzzy Mathematics in Economics and Engineering*, Physica-Verlag, Heidelberg, Germany, 2002.

[4] C. Duraisamy and B. Usha, "Another Approach to Solution of Fuzzy Differential Equations", *Applied Mathematical Sciences*, vol.4, No.16, 777-790, 2010.

[5] C. Duraisamy and B. Usha, "Numerical Solution of Fuzzy Differential Equations by Taylor method", *International journal of Mathematical Archive*, vol.2, 2011.

[6] R. Gethsi Sharmila and E. C. Henry Amirtharaj, "Numerical Solution of *Nth* - Order Fuzzy Initial Value Problems by Fourth Order Runge-Kutta Method Basesd On Centroidal Mean", *IOSR Journal of Mathematics*, vol. 6, no. 3, pp. 47-63, May - Jun. 2013.

[7] James J. Buckley and Thomas Feurihg, "Fuzzy Differential Equations", *Fuzzy Sets and Systems,* vol.110, pp.43-54, 2000.

[8] T. Jayakumar, K. Kanagarajan and S. Indrakumar, "Numerical Solution of Nth – Order Fuzzy Differential Equations by Runge Kutta

Method of Order Five", *International Journal of Mathematical Analysis*, vol. 6, no. 58, pp. 2885 - 2896, 2012.

[9] M. Ma, M. Friedman and A. kandel, "Numerical Solutions of Fuzzy Differential Equations", *Fuzzy Sets and Systems*, vol.105, 133-138, 1999.

[10] S. Sekar and A. Manonmani, "A study on time-varying singular non-linear systems via single-term Haar wavelet series", *International Review of Pure and Applied Mathematics*, vol.5, pp. 435-441, 2009.

[11] S. Sekar and G.Balaji, "Analysis of the differential equations on the sphere using single-term Haar wavelet series", *International Journal of Computer, Mathematical Sciences and Applications*, vol.4, pp.387-393, 2010.

[12] S. Sekar and M. Duraisamy, "A study on CNN based hole-filler template design using single-term Haar wavelet series", *International Journal of Logic Based Intelligent Systems*, vol.4, pp.17-26, 2010.

[13] S. Sekar and K. Jaganathan, "Analysis of the singular and stiff delay systems using single-term Haar wavelet series", *International Review of Pure and Applied Mathematics*, vol.6, pp. 135-142, 2010.

[14] S. Sekar and R. Kumar, "Numerical investigation of nonlinear volterra-hammerstein integral equations via single-term Haar wavelet series", *International Journal of Current Research*, vol.3, pp. 099-103, 2011.

[15] S. Sekar and E. Paramanathan, "A study on periodic and oscillatory problems using single-term Haar wavelet series", *International Journal of Current Research*, vol.2, pp. 097-105, 2011.

[16] S. Sekar and M. Vijayarakavan, "Analysis of the non-linear singular systems from fluid dynamics using single-term Haar wavelet series", *International Journal of Computer, Mathematical Sciences and Applications*, vol.4, pp. 15-29, 2010.